The Structure of Skew-Hamiltonian Matrices Revisited

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ABSTRACT

We give a different proof of W.C. Waterhouse's theorem: If char $\mathbb{F} \neq 2$ and $A \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian, then A is symplectically similar to $B \bigoplus B^{\top}$ for some $B \in M_n(\mathbb{F})$. His proof uses a structural idea drawn from the study of pairs of alternating forms, while our approach is based mainly on matrix theory. We also show that if char $\mathbb{F} \neq 2$, then every even-sized matrix over $\mathbb{F}$ is a sum of a symplectic and a Hamiltonian.

Keywords: symplectic, skew-Hamiltonian, Hamiltonian

INTRODUCTION

Let $M_{m,n}(\mathbb{F})$ denote the set of all m-by-n matrices over a field $\mathbb{F}$, and char $\mathbb{F}$ denote the characteristic of $\mathbb{F}$. We write $M_n(\mathbb{F})$ when $m = n$, and let $GL_n(\mathbb{F})$ denote the set of nonsingular matrices in $M_n(\mathbb{F})$. For a given symmetric or skew-symmetric $S \in GL_n(\mathbb{F})$, define the linear operator $\phi_S : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by $\phi_S(A) = S^{-1}A^{\top}S$, which is spectrum-preserving, involutory, and an antihomomorphism. Suppose $A \in M_n(\mathbb{F})$. We say that A is ϕ_S*-orthogonal* if A is nonsingular and $\phi_S(A) = A^{-1}$; A is ϕ_S*-symmetric* if $\phi_S(A) = A$; and A is ϕ_S*-skew-symmetric* if $\phi_S(A) = -A$ (Horn and Merino 1995). Let $J_{2n} := \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \in M_{2n}(\mathbb{F})$. If it is clear from the context, we denote J_{2n} by J. The ϕ_J-orthogonal, ϕ_J-symmetric, and ϕ_J-skew-symmetric matrices are

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also called *symplectic*, *skew-Hamiltonian*, and *Hamiltonian* matrices, respectively. If $Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2n}(\mathbb{F})$ is partitioned conformal to J, then

$$\phi_J(Y) = \begin{bmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{bmatrix}.$$

Hence, $X \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian if and only if $X = \begin{bmatrix} A & B \\ C & A^\top \end{bmatrix}$ where B and C are skew-symmetric; X is Hamiltonian if and only if $X = \begin{bmatrix} A & B \\ C & -A^\top \end{bmatrix}$ where B and C are symmetric; and X is symplectic if and only if $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that $AD^\top - BC^\top = I_n$, and $AB^\top$ and $CD^\top$ are symmetric.

Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in M_{2m}(\mathbb{F})$ and $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in M_{2n}(\mathbb{F})$, where each $A_j \in M_m(\mathbb{F})$ and $B_j \in M_n(\mathbb{F})$. The expanding sum of A and B is given by

$$A \boxplus B = \begin{bmatrix} A_1 \oplus B_1 & A_2 \oplus B_2 \\ A_3 \oplus B_3 & A_4 \oplus B_4 \end{bmatrix}.$$

If A and B are both symplectic, both skew-Hamiltonian, or both Hamiltonian, then $A \boxplus B$ is also symplectic, skew-Hamiltonian, or Hamiltonian, respectively.

In 1999, Faβbender et al. proved that every skew-Hamiltonian $A \in M_{2n}(\mathbb{R})$ is symplectically similar to $R \oplus R^\top$ for some $R \in M_n(\mathbb{R})$ [Theorem 1]. The proof uses Van Loan's reduction (Van Loan 1984) and solutions to Sylvester-like equations $BX - XB^\top = -C$ where B is 1-by-1 or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $(b \neq 0)$, and C is skew-symmetric. In 2001, Ikramov proved that every skew-Hamiltonian $A \in M_{2n}(\mathbb{C})$ is symplectically similar to $R \oplus R^\top$ for some $R \in M_n(\mathbb{C})$ [Theorem 3]. The proof proceeds by showing that $A = P^{-1}(B \oplus B^\top)P$, where B is in Jordan canonical form, and that $P \in GL_{2n}(\mathbb{C})$ has a ϕ_J polar decomposition. In 2005, Waterhouse proved that if $\mathbb{F}$ is a field of characteristic not 2, then every skew-Hamiltonian $A \in M_{2n}(\mathbb{F})$ is symplectically similar to $R \oplus R^\top$ for some $R \in M_n(\mathbb{F})$ [Theorem 3]. In the proof, $\mathbb{F}^{2n}$ is completely decomposed as an orthogonal direct sum of invariant subspaces, and a suitable change of (symplectic) basis is formed for each invariant subspace that gives rise to the desired matrix form. In this paper, we give a different proof of [Waterhouse 2005, Theorem 3]. We use the rational canonical form to obtain a basis $\mathcal{S}$ for an A-invariant totally degenerate subspace of $\mathbb{F}^{2n}$ with respect to the alternating bilinear form $\mathbf{q}(x,y) := x^\top J y$, and extend $\mathcal{S}$ to a symplectic basis $\mathcal{B}$ for $\mathbb{F}^{2n}$. If T is the matrix obtained from the columns of $\mathcal{B}$, then $T^{-1}AT$ is skew-Hamiltonian. We proceed to show that the latter is symplectically similar to a matrix of the desired form by (i) showing that if $C \in M_k(\mathbb{F})$ is a companion matrix, then for any skew-symmetric $D \in M_k(\mathbb{F})$, there is a symmetric X such that $CX - XC^\top = D$, and by (ii) using Hartwig (1983) [Corollary 2 of Section 4] which states that if $M = \begin{bmatrix} C(f) & D \\ 0 & E \end{bmatrix}$ where $C(f)$ is a companion matrix and $f(M) = 0$, then $C(f)X - XE = D$ has a solution.

Every even-sized complex matrix is a sum of a complex symplectic and a complex Hamiltonian [de la Cruz and Paras 2020, Theorem 1]. We show that if char $\mathbb{F} \neq 2$, then every even-sized matrix over $\mathbb{F}$ can be written as a sum of a symplectic and a Hamiltonian over $\mathbb{F}$.

PRELIMINARIES

For the sake of completeness, we include the following discussion on symplectic bases ([O'Meara 1978, Section 1.1] or [Kaplansky 1969, Section 1.9]). Throughout this section, $\mathbb{F}$ is a field with char $\mathbb{F} \neq 2$. We first show that if $B \in GL_n(\mathbb{F})$ is skew-symmetric, then n is even.

Lemma 1. *Let $S \in M_n(\mathbb{F})$ be skew-symmetric. Then there exists an $X \in GL_n(\mathbb{F})$ such that $S = X^\top (J_{2k} \oplus 0_{n-2k})X$ for some $k \geq 0$.*

Proof. Let $S = [s_{ij}] \in M_n(\mathbb{F})$ be skew-symmetric. We use induction on n. If $n = 1$, then $S = [0]$ and for any $a \in \mathbb{F} \setminus \{0\}$, we have $S = [a]^\top [0][a]$. Suppose $n > 1$ and the conclusion holds for skew-symmetric matrices of size smaller than n. If rank $S = 0$, then $S = 0_n$, and for any $X \in GL_n(\mathbb{F})$, we have $S = X^\top (0_n) X$. Suppose rank $S > 0$. Then there exist $i, j \in \{1,2,\dots,n\}$ with $i \neq j$ such that $-s_{ji} = s_{ij} \neq 0$. Let $X_1 \in GL_n(\mathbb{F})$ be a permutation matrix such that the $(1,2)$ and $(2,1)$ entries of $P := X_1^\top S X_1$ are nonzero. Partition $P = [p_{ij}]$ as $P = \begin{bmatrix} A & B \\ -B^\top & C \end{bmatrix}$ where $A = \begin{bmatrix} 0 & p_{12} \\ -p_{12} & 0 \end{bmatrix}$ with $p_{12} \neq 0$. Then A

is nonsingular and skew-symmetric. Take $X_2 = \begin{bmatrix} I_2 & -A^{-1}B \\ 0_{n-2,2} & I_{n-2} \end{bmatrix}$ so that

$$PX_2 = \begin{bmatrix} A & B \\ -B^\top & C \end{bmatrix}\begin{bmatrix} I_2 & -A^{-1}B \\ 0_{n-2,2} & I_{n-2} \end{bmatrix} = \begin{bmatrix} A & 0_{2,n-2} \\ -B^\top & B^\top A^{-1}B + C \end{bmatrix}$$

and

$$Q := X_2^\top P X_2 = \begin{bmatrix} I_2 & 0_{2,n-2} \\ B^\top A^{-1} & I_{n-2} \end{bmatrix}\begin{bmatrix} A & 0_{2,n-2} \\ -B^\top & B^\top A^{-1}B + C \end{bmatrix} = \begin{bmatrix} A & 0_{2,n-2} \\ 0_{n-2,2} & D \end{bmatrix}$$

where $D = B^\top A^{-1}B + C \in M_{n-2}(\mathbb{F})$ is skew-symmetric. By the induction hypothesis, there exists a $Y \in GL_{n-2}(\mathbb{F})$ such that $Y^\top DY = J_{2k} \oplus 0_{n-2k-2}$ for some $k \geq 0$. If we take $X_3 = \begin{bmatrix} p_{12}^{-1} & 0 \\ 0 & 1 \end{bmatrix} \oplus Y \in GL_n(\mathbb{F})$, then

$$\begin{aligned} R &:= X_3^\top Q X_3 \\ &= \left(\begin{bmatrix} p_{12}^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & p_{12} \\ -p_{12} & 0 \end{bmatrix}\begin{bmatrix} p_{12}^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right) \oplus (Y^\top DY) \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus J_{2k} \oplus 0_{n-2k-2}. \end{aligned}$$

Finally, there exists a permutation matrix $X_4 \in GL_n(\mathbb{F})$ such that $X_4^\top R X_4 = J_{2k+2} \oplus 0_{n-2k-2}$.

By Lemma 1, if $B \in GL_n(\mathbb{F})$ is skew-symmetric, then rank $B = n$ and there exists an $X \in GL_n(\mathbb{F})$ such that $B = X^\top J_{2k} X$ for some positive integer k. Thus, $n = 2k$.

A mapping $\mathbf{q}: \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ is said to be an *alternating bilinear form* if $\mathbf{q}$ is linear in both components and $\mathbf{q}(x,x) = 0$ for all $x \in \mathbb{F}^n$. If $x, y \in \mathbb{F}^n$, then $\mathbf{q}(x+y, x+y) = 0$, which implies $\mathbf{q}(x,y) = -\mathbf{q}(y,x)$. A subspace U of $\mathbb{F}^n$ with alternating bilinear form $\mathbf{q}$ is said to be *regular* if for all $0 \neq u \in U$, there exists a $v \in U$ such that $\mathbf{q}(u,v) \neq 0$; U is *totally degenerate* if $\mathbf{q}(U,U) = 0$; U is *maximal totally degenerate* if U is totally degenerate and for every totally degenerate subspace U_0 of $\mathbb{F}^n$ such that $U \subseteq U_0$, then $U_0 = U$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{F}^n$, and set $b_{ij} = \mathbf{q}(e_i, e_j)$. Let $B = [b_{ij}] \in M_n(\mathbb{F})$ and let $x, y \in \mathbb{F}^n$. Then $\mathbf{q}(x,y) = x^\top By$. Since $b_{ii} = \mathbf{q}(e_i, e_i) = 0$

and $b_{ij} = \mathbf{q}(e_i, e_j) = -\mathbf{q}(e_j, e_i) = -b_{ji}$, we get B is skew-symmetric. Thus, $\mathbb{F}^n$ is regular if and only if B is nonsingular. If B is nonsingular, then n is even by Lemma 1. If B is singular, then, by Lemma 1, there exist nonsingular $X \in GL_n(\mathbb{F})$ and an $A \in M_{n-1}(\mathbb{F})$ such that $X^\top BX = A \oplus [0]$. Thus, $e_n^\top X^\top B = e_n^\top (A \oplus [0])X^{-1} = 0_{1,n}$, so $\mathbf{q}(Xe_n, y) = 0$ for all $y \in \mathbb{F}^n$, but $Xe_n \neq 0$. That is, $\mathbb{F}^n$ is not regular when B is singular.

Let U be a subspace of a regular space $\mathbb{F}^{2n}$ with alternating bilinear form $\mathbf{q}(x, y) = x^\top By$. The *orthogonal complement* of U in $\mathbb{F}^{2n}$ is defined as

$$U^* = \{x \in \mathbb{F}^{2n} \mid \mathbf{q}(x, U) = 0\}.$$

If $u_1, \dots, u_k$ is a basis for U, we can view U^* as the kernel of the surjective linear transformation $f: \mathbb{F}^{2n} \to \mathbb{F}^k$ defined by $f(x) = [u_1 \cdots u_k]^\top Bx$. Hence $\dim U + \dim U^* = 2n$. If U is totally degenerate, then $U \subseteq U^*$, so $\dim U \leq \dim U^* = 2n - \dim U$, and $\dim U \leq n$.

Lemma 2. *Suppose $\mathbb{F}^{2n}$ is regular. Let U be a totally degenerate subspace of $\mathbb{F}^{2n}$ with basis $\{x_1, \dots, x_r\}$. If $r < n$, then there exists an $x_{r+1} \in \mathbb{F}^{2n}$ such that $\{x_1, \dots, x_r, x_{r+1}\}$ is linearly independent and $U_1 = \operatorname{span}\{x_1, \dots, x_r, x_{r+1}\}$ is totally degenerate.*

Proof. Suppose $\mathbb{F}^{2n}$ is regular. Let U be a totally degenerate subspace of $\mathbb{F}^{2n}$ with basis $\{x_1, \dots, x_r\}$. If $r < n$, then $\dim(U^*) = 2n - r > n$ and $U^* \setminus U$ is non-empty. Let $x_{r+1} \in U^* \setminus U$. Then $\{x_1, \dots, x_r, x_{r+1}\}$ is linearly independent since $x_{r+1} \notin U = \operatorname{span}\{x_1, \dots, x_r\}$ and $\{x_1, \dots, x_r\}$ is linearly independent. Since U is totally degenerate and $x_{r+1} \in U^*$, we get $U_1 = \operatorname{span}\{x_1, \dots, x_r, x_{r+1}\}$ is totally degenerate.

We say that $\mathbb{F}^{2n}$ has the *orthogonal splitting*

$$\mathbb{F}^{2n} = V_1 \perp V_2 \perp \cdots \perp V_r$$

into subspaces $V_1, \dots, V_r$ if $\mathbb{F}^{2n} = V_1 \oplus \cdots \oplus V_r$ and $\mathbf{q}(V_i, V_j) = 0$ whenever $i \neq j$.

Lemma 3. *Let* U *and* W *be subspaces of a regular space* $\mathbb{F}^{2n}$. *If* U *is regular, then* $\mathbb{F}^{2n} = U \perp U^*$. *If* $\mathbb{F}^{2n} = U \perp W$, *then* U *and* W *are regular.*

Proof. Let U and W be subspaces of a regular space $\mathbb{F}^{2n}$. If U is regular, then $U \cap U^* = \{0\}$, so $\mathbb{F}^{2n} = U \bigoplus U^*$, which implies $\mathbb{F}^{2n} = U \perp U^*$. If $\mathbb{F}^{2n} = U \perp W$, then $\{0\} = (\mathbb{F}^{2n})^* = U^* + W^*$, so $U^* = W^* = \{0\}$, which implies U and W are regular.

We say that $\mathcal{B} = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a *symplectic basis* for $\mathbb{F}^{2n}$ with respect to the alternating bilinear form $\mathbf{q}$ if $\mathcal{B}$ is a basis for $\mathbb{F}^{2n}$ such that $\mathbf{q}(u_i, u_j) = 0 = \mathbf{q}(v_i, v_j)$, and $\mathbf{q}(u_i, v_j) = \delta_{ij}$ for all i, j.

Lemma 4. *If* U *is a maximal totally degenerate subspace of a regular* $\mathbb{F}^{2n}$ *with basis* $\{x_1, \ldots, x_n\}$, *then there exist* $y_1, \ldots, y_n \in \mathbb{F}^{2n}$ *such that* $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ *is a symplectic basis for* $\mathbb{F}^{2n}$.

Proof. Let U be a maximal totally degenerate subspace of a regular $\mathbb{F}^{2n}$ with basis $\{x_1, \ldots, x_n\}$. There exists $y_1 \in \{x_2, \ldots, x_n\}^*$ such that $\mathbf{q}(x_1, y_1) \neq 0$, since otherwise span$(U \cup \{y_1\})$ is totally degenerate. Without loss of generality, $\mathbf{q}$ $(x_1, y_1) = 1$. Let $P_1 = \text{span}\{x_1, y_1\}$, which is regular. Then $\mathbb{F}^{2n} = P_1 \perp P_1^*$ and $\{x_2, \ldots, x_n\} \subseteq P_1^*$. Since $\mathbb{F}^{2n}$ is regular, we have P_1^* is regular. We can continue to obtain vectors $y_2, \ldots, y_n$ such that $\mathbf{q}(x_k, y_k) = 1$, $P_k = \text{span}\{x_k, y_k\}$ is regular, and $\mathbb{F}^{2n} = P_1 \perp P_2 \perp \cdots \perp P_n$. Therefore, $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is a symplectic basis for $\mathbb{F}^{2n}$.

Let $X \in M_{2n}(\mathbb{F})$ and x_i be the ith column of X. Then X is symplectic if and only if

$$\begin{bmatrix} x_1^\top J x_1 & \cdots & x_1^\top J x_n & x_1^\top J x_{n+1} & \cdots & x_1^\top J x_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_n^\top J x_1 & \cdots & x_n^\top J x_n & x_n^\top J x_{n+1} & \cdots & x_n^\top J x_{2n} \\ x_{n+1}^\top J x_1 & \cdots & x_{n+1}^\top J x_n & x_{n+1}^\top J x_{n+1} & \cdots & x_{n+1}^\top J x_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{2n}^\top J x_1 & \cdots & x_{2n}^\top J x_n & x_{2n}^\top J x_{n+1} & \cdots & x_{2n}^\top J x_{2n} \end{bmatrix} = X^\top J X = J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},$$

i.e., the columns of X form a symplectic basis for $\mathbb{F}^{2n}$ with respect to the alternating bilinear form

$$\mathbf{q}(x, y) = x^{\top} J y. \tag{1}$$

If $A \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian, we use the rational canonical form to obtain a basis $\mathcal{S}$ for an A-invariant totally degenerate subspace of $\mathbb{F}^{2n}$ with respect to the alternating bilinear form in (1), and extend $\mathcal{S}$ to a symplectic basis $\mathcal{B}$ for $\mathbb{F}^{2n}$. If $T \in M_{2n}(\mathbb{F})$ such that the ith column of T is the ith element of $\mathcal{B}$, then T is symplectic, and A is symplectically similar to $T^{-1}AT$.

MAIN RESULTS

Suppose char $\mathbb{F} = 2$ and $\lambda \in \mathbb{F}$. Let $E_{ij} \in M_n(\mathbb{F})$ be such that all entries are 0 except for a 1 in the (i, j)-entry. Let $J_n(\lambda) \in M_n(\mathbb{F})$ which has λ along the main diagonal, 1 along the superdiagonal and 0 elsewhere. Then

$$A(\lambda) = \begin{bmatrix} J_n(\lambda) & E_{nn} \\ 0_n & J_n(\lambda)^{\top} \end{bmatrix} \tag{2}$$

is skew-Hamiltonian and its Jordan canonical form is $J_{2n}(\lambda)$. Hence, $A(\lambda)$ is not similar to a direct sum of two matrices.

Suppose char $\mathbb{F} \neq 2$ and $A \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian. Let $m_A = x^k + \sum_{i=0}^{k-1} a_i x^i$ be the minimal polynomial of A. Then the *companion matrix* of m_A is given by

$$C(m_A) = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ & & & -a_1 \\ & I_{k-1} & & \vdots \\ & & & -a_{k-1} \end{bmatrix}. \tag{3}$$

The rational canonical form theorem guarantees that there exists an $X \in GL_{2n}(\mathbb{F})$ such that

$$AX = X \begin{bmatrix} C(m_A) & 0_{k,2n-k} \\ 0_{2n-k,k} & G \end{bmatrix} \tag{4}$$

for some $G \in M_{2n-k}(\mathbb{F})$. Let v_i be the ith column of X. Then for $1 \leq i \leq k-1$, we have $Av_i = v_{i+i}$. Thus for $2 \leq i \leq k$, we have $v_i = A^{i-1}v_1$. Let $v = v_1$. Then the set $\mathcal{S} = \{v, Av, \ldots, A^{k-1}v\}$ is linearly independent since X is nonsingular. Since A is skew-Hamiltonian, if $\mathbf{q}$ is the alternating bilinear form in (1), then $\mathbf{q}(x, Ay) = \mathbf{q}(Ax, y)$ for all $x, y \in \mathbb{F}^{2n}$. Hence $\mathbf{q}(A^i v, A^j v) = \mathbf{q}(A^j v, A^i v) = -\mathbf{q}(A^i v, A^j v)$ for all nonnegative integers i, j. Since char $\mathbb{F} \neq 2$, we have $\mathbf{q}(A^i v, A^j v) = 0$ for all i, j. Hence $U :=$ span $\mathcal{S}$ is totally degenerate.

If $k = n$, then $\mathcal{S}$ is a basis for the maximal totally degenerate subspace U of $\mathbb{F}^{2n}$. If $k < n$, then, by applying Lemma 2 $(n-k)$ times, we can extend $\mathcal{S}$ to a basis

$$\mathcal{S}_1 = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

for a maximal totally degenerate subspace U_1 of $\mathbb{F}^{2n}$. By Lemma 4, we can extend $\mathcal{S}_1$ to a symplectic basis

$$\mathcal{B} = \{v_1, \dots, v_n, y_1, \dots, y_n\}$$

for $\mathbb{F}^{2n}$. Let $Q \in M_{2n}(\mathbb{F})$ be such that its columns are the elements of $\mathcal{B}$. Then Q is symplectic and

$$Q^{-1}AQ = \begin{bmatrix} C(m_A) & A_1 & B_1 & B_2 \\ 0 & A_2 & B_3 & B_4 \\ 0 & D_1 & C_1 & C_2 \\ 0 & D_2 & C_3 & C_4 \end{bmatrix}. \qquad (5)$$

Since $Q^{-1}AQ$ is skew-Hamiltonian, $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ and $\begin{bmatrix} 0 & D_1 \\ 0 & D_2 \end{bmatrix}$ are skew-symmetric, and $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ is the transpose of $\begin{bmatrix} C(m_A) & A_1 \\ 0 & A_2 \end{bmatrix}$. Hence,

$$Q^{-1}AQ = \begin{bmatrix} C(m_A) & A_1 & B_1 & B_2 \\ 0 & A_2 & -B_2^{\mathsf{T}} & B_4 \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & D_2 & A_1^{\mathsf{T}} & A_2^{\mathsf{T}} \end{bmatrix} \qquad (6)$$

where B_1, B_4, D_2 are skew-symmetric.

Lemma 5. *Suppose char* $\mathbb{F} \neq 2$. *Let* $A \in M_{2n}(\mathbb{F})$ *be skew-Hamiltonian and let* m_A *be the minimal polynomial of* A. *There exists a symplectic* $Q \in M_{2n}(\mathbb{F})$ *such that*

$$Q^{-1}AQ = \begin{bmatrix} C(m_A) & A_1 & B_1 & B_2 \\ 0 & A_2 & -B_2^{\top} & B_3 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & D & A_1^{\top} & A_2^{\top} \end{bmatrix} \tag{7}$$

where B_1, B_3, *and* D *are skew-symmetric.*

The following lemma is used to construct symplectic matrices that give rise to some desired forms under similarity.

Lemma 6. *Suppose char* $\mathbb{F} \neq 2$ *and* $p \in \mathbb{F}[x]$ *is monic of positive degree* n. *If* $B \in M_n(\mathbb{F})$ *is skew-symmetric, then there is a symmetric* $Q \in M_n(\mathbb{F})$ *such that* $C(p)Q - QC(p)^{\top} = B$.

Proof. Suppose char $\mathbb{F} \neq 2$ and $p = \sum_{i=0}^{n} a_i x^i \in \mathbb{F}[x]$ is monic with positive degree n. If $n = 1$ and $B \in M_1(\mathbb{F})$ is skew-symmetric, then $B = [0]$, and any $Q \in M_1(\mathbb{F})$ is symmetric and satisfies $C(p)Q - QC(p)^{\top} = B$.

Suppose $n > 1$. Let $\mathcal{S}_n(\mathbb{F})$ and $\mathcal{K}_n(\mathbb{F})$ respectively denote the set of symmetric matrices and skew-symmetric matrices in $M_n(\mathbb{F})$, and let $\gamma : \mathcal{S}_n(\mathbb{F}) \to \mathcal{K}_n(\mathbb{F})$ be the linear transformation defined by

$$\gamma(X) = C(p)X - XC(p)^{\top}.$$

We prove the lemma by showing that γ is onto. A basis for $\mathcal{S}_n(\mathbb{F})$ is

$$B_S = \{E_{ii}, E_{ij} + E_{ji} \mid i, j = 1, \ldots, n \text{ and } i < j\},$$

while a basis for $\mathcal{K}_n(\mathbb{F})$ is

$$B_K = \{E_{ij} - E_{ji} \mid i, j = 1, \ldots, n \text{ and } i < j\}.$$

We show that $B_K \subset \operatorname{span} \gamma(B_S)$. For $i < n$, notice that

$$\begin{aligned} (-1)\gamma(E_{ii}) &= -C(p)E_{ii} + E_{ii}C(p)^{\top} \\ &= \begin{bmatrix} 0 & \cdots & 0 & a_0 \\ -1 & & & a_1 \\ & \ddots & & \vdots \\ & & -1 & a_{n-1} \end{bmatrix} E_{ii} + E_{ii} \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \\ &= -E_{i+1,i} + E_{i,i+1}. \end{aligned}$$

If $j > i+1$, then $j-1 < n$ and

$$-\frac{1}{2}\sum_{k=0}^{j-i-1} \gamma\left(E_{i+k,j-k-1} + E_{j-k-1,i+k}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{j-i-1}\left[-C(p)\left(E_{i+k,j-k-1} + E_{j-k-1,i+k}\right) + \left(E_{i+k,j-k-1} + E_{j-k-1,i+k}\right)C(p)^{\mathsf{T}}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{j-i-1}\left[-E_{i+k+1,j-k-1} - E_{j-k,i+k} + E_{i+k,j-k} + E_{j-k-1,i+k+1}\right]$$

$$= -\frac{1}{2}\sum_{k=0}^{j-i-1} E_{i+k+1,j-k-1} + \frac{1}{2}\sum_{k=0}^{j-i-1} E_{i+k,j-k} - \frac{1}{2}\sum_{k=0}^{j-i-1} E_{j-k,i+k} + \frac{1}{2}\sum_{k=0}^{j-i-1} E_{j-k-1,i+k+1}$$

$$= -\frac{1}{2}E_{ji} + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} + \frac{1}{2}E_{ij}$$

$$= E_{ij} - E_{ji}.$$

The following theorem gives a characterization of a skew-Hamiltonian, up to symplectic similarity.

Theorem 7. *Every skew-Hamiltonian* $A \in M_{2n}(\mathbb{F})$ *is symplectically similar to* $Y \oplus Y^{\mathsf{T}}$ *for some* $Y \in M_n(\mathbb{F})$ *if and only if char* $\mathbb{F} \neq 2$.

Proof. It remains to prove sufficiency. Let char $\mathbb{F} \neq 2$ and $A \in M_{2n}(\mathbb{F})$ be skew-Hamiltonian. We use induction on n. If $n = 1$, then $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some $a \in \mathbb{F}$. Suppose $n > 1$ and the conclusion holds for skew-Hamiltonian matrices of size smaller than $2n$. Let $k = \deg m_A$. If $k = n$, then, by Lemma 5, A is symplectically similar to

$$X = \begin{bmatrix} C(m_A) & B \\ 0 & C(m_A)^{\mathsf{T}} \end{bmatrix}, \tag{8}$$

for some skew-symmetric B. By Lemma 6, there exists a symmetric Q such that $C(m_A)Q - QC(m_A)^{\mathsf{T}} = B$. Set $Z = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}$ so that $Z^{-1} = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$ and

$$X_1 = Z^{-1}XZ = C(m_A) \oplus C(m_A)^{\mathsf{T}}. \tag{9}$$

Suppose $k < n$. By Lemma 5, A is symplectically similar to

$$X_2 = \begin{bmatrix} C(m_A) & A_1 & B_1 & B_2 \\ 0 & A_2 & -B_2^{\mathsf{T}} & B_3 \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & D & A_1^{\mathsf{T}} & A_2^{\mathsf{T}} \end{bmatrix}, \tag{10}$$

for some skew-symmetric B_1, B_3, D. Let $G = \begin{bmatrix} A_2 & B_3 \\ D & A_2^{\mathsf{T}} \end{bmatrix}$, which is skew-Hamiltonian. By the induction hypothesis, there exists a symplectic P such that $P^{-1}GP = E \oplus E^{\mathsf{T}}$ for some E. Partition $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ conformal to G. Then $I_{2k} \boxplus P$ is symplectic and X_2 is symplectically similar to

$$X_3 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & P_4^{\mathsf{T}} & 0 & -P_2^{\mathsf{T}} \\ 0 & 0 & I & 0 \\ 0 & -P_3^{\mathsf{T}} & 0 & P_1^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} C(m_A) & A_1 & B_1 & B_2 \\ 0 & A_2 & -B_2^{\mathsf{T}} & B_3 \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & D & A_1^{\mathsf{T}} & A_2^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & P_1 & 0 & P_2 \\ 0 & 0 & I & 0 \\ 0 & P_3 & 0 & P_4 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & P_4^{\mathsf{T}} & 0 & -P_2^{\mathsf{T}} \\ 0 & 0 & I & 0 \\ 0 & -P_3^{\mathsf{T}} & 0 & P_1^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} C(m_A) & A_1P_1 + B_2P_3 & B_1 & A_1P_2 + B_2P_4 \\ 0 & A_2P_1 + B_3P_3 & -B_2^{\mathsf{T}} & A_2P_2 + B_3P_4 \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & DP_1 + A_2^{\mathsf{T}}P_3 & A_1^{\mathsf{T}} & DP_2 + A_2^{\mathsf{T}}P_4 \end{bmatrix}$$

$$= \begin{bmatrix} C(m_A) & A_1P_1 + B_2P_3 & B_1 & A_1P_2 + B_2P_4 \\ 0 & P_4^{\mathsf{T}}(A_2P_1 + B_3P_3) - P_2^{\mathsf{T}}(DP_1 + A_2^{\mathsf{T}}P_3) & -P_4^{\mathsf{T}}B_2^{\mathsf{T}} - P_2^{\mathsf{T}}A_1^{\mathsf{T}} & P_4^{\mathsf{T}}(A_2P_2 + B_3P_4) - P_2^{\mathsf{T}}(DP_2 + A_2^{\mathsf{T}}P_4) \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & -P_3^{\mathsf{T}}(A_2P_1 + B_3P_3) + P_1^{\mathsf{T}}(DP_1 + A_2^{\mathsf{T}}P_3) & P_3^{\mathsf{T}}B_2^{\mathsf{T}} + P_1^{\mathsf{T}}A_1^{\mathsf{T}} & -P_3^{\mathsf{T}}(A_2P_2 + B_3P_4) + P_1^{\mathsf{T}}(DP_2 + A_2^{\mathsf{T}}P_4) \end{bmatrix}$$

$$= \begin{bmatrix} C(m_A) & A_3 & B_1 & B_4 \\ 0 & E & -B_4^{\mathsf{T}} & 0 \\ 0 & 0 & C(m_A)^{\mathsf{T}} & 0 \\ 0 & 0 & A_3^{\mathsf{T}} & E^{\mathsf{T}} \end{bmatrix},$$

for some $A_3, B_4 \in M_{k,n-k}(\mathbb{F})$. Let $H = \begin{bmatrix} C(m_A) & A_3 \\ 0 & E \end{bmatrix}$. Because $m_A(A) = 0$, we have $m_A(H) = 0$. Corollary 2 Section 4 of Hartwig (1983) guarantees that there exists an $R \in M_{k,n-k}(\mathbb{F})$ such that $C(m_A)R - RE = A_3$. Notice that X_3 is symplectically similar to

$$\begin{aligned}
X_4 &= \begin{bmatrix} I & R & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -R^{\top} & I \end{bmatrix} \begin{bmatrix} C(m_A) & A_3 & B_1 & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & A_3^{\top} & E^{\top} \end{bmatrix} \begin{bmatrix} I & -R & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & R^{\top} & I \end{bmatrix} \\
&= \begin{bmatrix} I & R & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -R^{\top} & I \end{bmatrix} \begin{bmatrix} C(m_A) & -C(m_A)R + A_3 & B_1 + B_4R^{\top} & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & A_3^{\top} + E^{\top}R^{\top} & E^{\top} \end{bmatrix} \\
&= \begin{bmatrix} C(m_A) & -C(m_A)R + A_3 + RE & B_1 + B_4R^{\top} - RB_4^{\top} & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & A_3^{\top} + E^{\top}R^{\top} - R^{\top}C(m_A)^{\top} & E^{\top} \end{bmatrix} \\
&= \begin{bmatrix} C(m_A) & 0 & B_5 & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & 0 & E^{\top} \end{bmatrix},
\end{aligned}$$

for some skew-symmetric $B_5 \in M_k(\mathbb{F})$. Lemma 6 ensures that there exists a symmetric $S \in M_k(\mathbb{F})$ such that $C(m_A)S - SC(m_A)^{\top} = B_5$. Hence, X_4 is symplectically similar to

$$\begin{aligned}
X_5 &= \begin{bmatrix} I & 0 & S & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} C(m_A) & 0 & B_5 & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & 0 & E^{\top} \end{bmatrix} \begin{bmatrix} I & 0 & -S & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \\
&= \begin{bmatrix} I & 0 & S & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} C(m_A) & 0 & -C(m_A)S + B_5 & B_4 \\ 0 & E & -B_4^{\top} & 0 \\ 0 & 0 & C(m_A)^{\top} & 0 \\ 0 & 0 & 0 & E^{\top} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} C(m_A) & 0 & -C(m_A)S + B_5 + SC(m_A)^\top & B_4 \\ 0 & E & -B_4^\top & 0 \\ 0 & 0 & C(m_A)^\top & 0 \\ 0 & 0 & 0 & E^\top \end{bmatrix}$$

$$= \begin{bmatrix} C(m_A) & 0 & 0 & B_4 \\ 0 & E & -B_4^\top & 0 \\ 0 & 0 & C(m_A)^\top & 0 \\ 0 & 0 & 0 & E^\top \end{bmatrix}.$$

Finally, $I_{2k} \boxplus J_{2(n-k)}$ is symplectic and X_5 is symplectically similar to

$$X_6 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m_A) & 0 & 0 & B_4 \\ 0 & E & -B_4^\top & 0 \\ 0 & 0 & C(m_A)^\top & 0 \\ 0 & 0 & 0 & E^\top \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m_A) & -B_4 & 0 & 0 \\ 0 & 0 & -B_4^\top & E \\ 0 & 0 & C(m_A)^\top & 0 \\ 0 & -E^\top & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C(m_A) & -B_4 & 0 & 0 \\ 0 & E^\top & 0 & 0 \\ 0 & 0 & C(m_A)^\top & 0 \\ 0 & 0 & -B_4^\top & E \end{bmatrix}$$

which is a matrix of the form $Y \oplus Y^\top$.

Suppose char $\mathbb{F} \neq 2$. If $A \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian, then, by Theorem 7, there exist a symplectic $P \in M_{2n}(\mathbb{F})$ and an $X \in M_n(\mathbb{F})$ such that $P^{-1}AP = X \oplus X^\top$. If R is the rational canonical form of X, then there exists a $Q \in GL_n(\mathbb{F})$ such that $Q^{-1}XQ = R$. Note that $Q \oplus Q^{-\top}$ is symplectic and

$$(Q^{-1} \oplus Q^\top)(P^{-1}AP)(Q \oplus Q^{-\top}) = (Q^{-1}XQ) \oplus (Q^\top X^\top Q^{-\top}) = R \oplus R^\top.$$

Hence $S_1 = P(Q \oplus Q^{-\top})$ is symplectic and $S_1^{-1}AS_1 = R \oplus R^\top$. Suppose $B \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian and similar to A. Then there exists a symplectic $S_2 \in M_{2n}(\mathbb{F})$ such that $S_2BS_2^{-1} = R \oplus R^\top$. Moreover, S_1S_2 is symplectic and

$$S_2^{-1}S_1^{-1}AS_1S_2 = S_2^{-1}(R \oplus R^{\mathrm{T}})S_2 = B.$$

Corollary 8. *Let char* $\mathbb{F} \neq 2$ *and* $A, B \in M_{2n}(\mathbb{F})$ *be skew-Hamiltonian. Then* A *and* B *are similar if and only if they are symplectically similar.*

Let char $\mathbb{F} \neq 2$ and $S \in GL_{2n}(\mathbb{F})$ be skew-symmetric. Then there is an $X \in GL_{2n}(\mathbb{F})$ such that $S = X^{\mathrm{T}} J_{2n} X$. Since $\phi_S(D) = X^{-1}\phi_J(XDX^{-1})X$ for all $D \in M_{2n}(\mathbb{F})$, if $A, B \in M_{2n}(\mathbb{F})$ are ϕ_S-symmetric and similar, then XAX^{-1} and XBX^{-1} are ϕ_J-symmetric and similar. Corollary 8 guarantees that there exists a symplectic $P \in M_{2n}(\mathbb{F})$ such that $P^{-1}(XAX^{-1})P = XBX^{-1}$. Set $Q = X^{-1}PX$. Then Q is ϕ_S-orthogonal, and $Q^{-1}AQ = B$.

Corollary 9. *Let char* $\mathbb{F} \neq 2$ *and* $S \in GL_{2n}(\mathbb{F})$ *be skew-symmetric. Let* $A, B \in M_{2n}(\mathbb{F})$ *be* ϕ_S*-symmetric. If* A *and* B *are similar, then the matrix of similarity may be taken to be* ϕ_S*-orthogonal.*

Let $\mathbb{F}$ be a field and $A \in M_{2n}(\mathbb{F})$. If $A = X + H$ for some symplectic $X \in M_{2n}(\mathbb{F})$ and Hamiltonian $H \in M_{2n}(\mathbb{F})$, then

$$A + \phi_J(A) = X + H + \phi_J(X) - H = X + \phi_J(X).$$

Conversely, if $A + \phi_J(A) = Y + \phi_J(Y)$ for some symplectic $Y \in M_{2n}(\mathbb{F})$, then $\phi_J(Y) - \phi_J(A) = A - Y$ leading to

$$\phi_J[\phi_J(Y) - \phi_J(A)] = \phi_J(A - Y) = \phi_J(A) - \phi_J(Y) = -[\phi_J(Y) - \phi_J(A)].$$

Thus, $\phi_J(Y) - \phi_J(A)$ is Hamiltonian and $A = Y + [\phi_J(Y) - \phi_J(A)]$ is a sum of a symplectic and a Hamiltonian.

Lemma 10. *Let* $\mathbb{F}$ *be a field and* $A \in M_{2n}(\mathbb{F})$*. Then* $A = X + H$ *for some symplectic* $X \in M_{2n}(\mathbb{F})$ *and Hamiltonian* $H \in M_{2n}(\mathbb{F})$ *if and only if* $A + \phi_J(A) = Y + \phi_J(Y)$ *for some symplectic* $Y \in M_{2n}(\mathbb{F})$*.*

Thus, to prove that an $A \in M_{2n}(\mathbb{F})$ is a sum of a symplectic and a Hamiltonian, it is enough to show that $A + \phi_J(A) = W + \phi_J(W)$ for some symplectic $W \in M_{2n}(\mathbb{F})$.

Theorem 11. *Suppose char* $\mathbb{F} \neq 2$. *If* $A \in M_{2n}(\mathbb{F})$ *is skew-Hamiltonian, then there exists a symplectic* $X \in M_{2n}(\mathbb{F})$ *such that* $A = X + \phi_J(X)$.

Proof. Suppose char $\mathbb{F} \neq 2$ and $A \in M_{2n}(\mathbb{F})$ is skew-Hamiltonian. By Theorem 7, there exists a symplectic $P \in M_{2n}(\mathbb{F})$ such that

$$P^{-1}AP = G \bigoplus G^{\top}$$

for some $G \in M_n(\mathbb{F})$. There exists a symmetric $S \in GL_n(\mathbb{F})$ such that $G^{\top} = S^{-1}GS$ ([Frobenius 1910 or Kaplansky 1969, Theorem 66]). Set $X := P\begin{bmatrix} 0_n & S \\ -S^{-1} & G^{\top} \end{bmatrix}P^{-1}$, and notice that

$$\phi_J(X)X = P\begin{bmatrix} G & -S \\ S^{-1} & 0_n \end{bmatrix}\begin{bmatrix} 0_n & S \\ -S^{-1} & G^{\top} \end{bmatrix}P^{-1} = P\begin{bmatrix} I_n & GS - SG^{\top} \\ 0 & I_n \end{bmatrix}P^{-1} = I_{2n},$$

i.e., X is symplectic. Moreover,

$$X + \phi_J(X) = P\begin{bmatrix} 0_n & S \\ -S^{-1} & G^{\top} \end{bmatrix}P^{-1} + P\begin{bmatrix} G & -S \\ S^{-1} & 0_n \end{bmatrix}P^{-1} = P(G \bigoplus G^{\top})P^{-1} = A.$$

Corollary 12. *Suppose char* $\mathbb{F} \neq 2$. *If* $A \in M_{2n}(\mathbb{F})$, *then there exist a symplectic* $X \in M_{2n}(\mathbb{F})$ *and a Hamiltonian* $H \in M_{2n}(\mathbb{F})$ *such that* $A = X + H$.

The preceding conclusion does not hold when char $\mathbb{F} = 2$. Suppose char $\mathbb{F} = 2$ and $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{F})$, which is skew-Hamiltonian. Suppose there exists a symplectic $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{F})$ such that $A = X + \phi_J(X)$. Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A = X + \phi_J(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (a+d)I_2,$$

which is a contradiction.

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